Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
International
Advanced Level

Centre Number	Candidate Number

Friday 8 January 2021

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Lilac), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

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Pearson

1. (a) Show that the equation $4x - 2\sin x - 1 = 0$, where x is in radians, has a root α in the interval $[0.2, 0.6]$

(2)

(b) Starting with the interval $[0.2, 0.6]$, use interval bisection twice to find an interval of width 0.1 in which α lies.

(3)

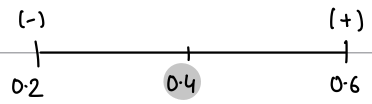
a. $f(x) = 4x - 2\sin x - 1$

$f(0.2) = 4(0.2) - 2\sin(0.2) - 1 = -0.5973386616$ $\quad f(0.2) < 0$

$f(0.6) = 4(0.6) - 2\sin(0.6) - 1 = 0.2707150532$ $\quad f(0.6) > 0$

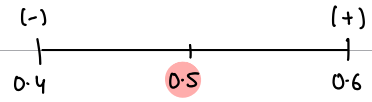
There is a sign change from negative to positive and $f(x)$ is continuous, $\therefore$ root, α, lies in the interval $[0.2, 0.6]$.

b. 1st interval bisection: $\dfrac{0.2 + 0.6}{2} = 0.4$

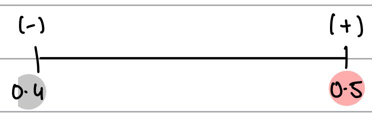


$f(0.4) = -0.1788366846$

2nd interval bisection: $\dfrac{0.4 + 0.6}{2} = 0.5$



$f(0.5) = 0.04114892279$



$\boxed{0.4 \leq \alpha \leq 0.5}$

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2. Given that $x = \frac{3}{8} + \frac{\sqrt{71}}{8}\text{i}$ is a root of the equation

$$4x^3 - 19x^2 + px + q = 0$$

(a) write down the other complex root of the equation.

(1)

Given that $x = 4$ is also a root of the equation,

(b) find the value of p and the value of q.

(4)

a. For a cubic eq^n with form: $ax^3 + bx^2 + cx + d$, there are 3 roots.
There are 2 possibilities: 3 real roots
1 real root and 2 complex roots (conjugate pair)

The other complex root must be the complex conjugate of $\left(\frac{3}{8} + \frac{\sqrt{71}}{8}i\right)$
(flip sign of imaginary component

$\boxed{\frac{3}{8} - \frac{\sqrt{71}}{8}i}$

b. $\left(x - \left(\frac{3}{8} - \frac{\sqrt{71}}{8}i\right)\right)\left(x - \left(\frac{3}{8} + \frac{\sqrt{71}}{8}i\right)\right)(x-4) = 0$

$\left(x - \frac{3}{8} + \frac{\sqrt{71}}{8}i\right)\left(x - \frac{3}{8} - \frac{\sqrt{71}}{8}i\right)(x-4) = 0$

$\left(x^2 - \frac{3}{8}x - \cancel{\frac{\sqrt{71}}{8}ix} - \frac{3}{8}x + \frac{9}{64} + \cancel{\frac{3\sqrt{71}}{64}i} + \cancel{\frac{\sqrt{71}}{8}ix} - \cancel{\frac{3\sqrt{71}}{64}i} - \frac{71}{64}i^2\right)(x-4) = 0$

$\left(x^2 - \frac{6}{8}x + \frac{9}{64} - \frac{71}{64}(-1)\right)(x-4) = 0$

$\left(x^2 - \frac{6}{8}x + \frac{80}{64}\right)(x-4) = 0$

$\left(x^2 - \frac{3}{4}x + \frac{5}{4}\right)(x-4) = 0$

$x^3 - \frac{3}{4}x^2 + \frac{5}{4}x - 4x^2 + 3x - 5 = 0$

$x^3 - \frac{19}{4}x^2 + \frac{17}{4}x - 5 = 0$ $\quad \times 4$

$4x^3 - 19x^2 + 17x - 20 = 0$

Compare coefficients with eq^n above: $p = 17$, $q = -20$

$\boxed{p = 17,\ q = -20}$

3. The matrix **M** is defined by

$$\mathbf{M} = \begin{pmatrix} k+5 & -2 \\ -3 & k \end{pmatrix}$$

(a) Determine the values of k for which **M** is singular.

(2)

Given that **M** is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of k.

(2)

a. Matrix M is singular $\Rightarrow$ $\det(M) = 0$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(x) = (a)(d) - (b)(c)$

$\det(M) = (k+5)(k) - (-2)(-3)$

$= (k^2 + 5k) - (6)$

$= k^2 + 5k - 6$

$\det(M) = k^2 + 5k - 6$

Set $\det(m) = 0$ $\Rightarrow$ $k^2 + 5k - 6 = 0$

$(k+6)(k-1) = 0$

$k = -6$ or $k = 1$

$\boxed{k = -6 \text{ or } k = 1}$

b. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \dfrac{1}{\det(x)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(-2) = 2$

$-c = -(-3) = 3$

$\boxed{M^{-1} = \dfrac{1}{k^2 + 5k + 6} \begin{bmatrix} k & 2 \\ 3 & k+5 \end{bmatrix}}$

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4. The equation $2x^2 + 5x + 7 = 0$ has roots α and β

Without solving the equation

(a) determine the exact value of $\alpha^3 + \beta^3$

(3)

(b) form a quadratic equation, with integer coefficients, which has roots

$$\frac{\alpha^2}{\beta} \text{ and } \frac{\beta^2}{\alpha}$$

(5)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$2x^2 + 5x + 7 = 0$ $\div 2$

$x^2 + \frac{5}{2}x + \frac{7}{2} = 0$

$\alpha + \beta = -\frac{5}{2}$

$\alpha\beta = \frac{7}{2}$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \left(-\frac{5}{2}\right)^3 - 3\left(\frac{7}{2}\right)\left(-\frac{5}{2}\right)$

$= \frac{85}{8}$

$\boxed{\alpha^3 + \beta^3 = \frac{85}{8}}$

b. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - \left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right)x + \left(\frac{\alpha^{\cancel{2}}}{\cancel{\beta}} \times \frac{\beta^{\cancel{2}}}{\cancel{\alpha}}\right) = 0$

$x^2 - \left(\frac{\alpha^3 + \beta^3}{\alpha\beta}\right)x + (\alpha\beta) = 0$

use ans from part (a)

$x^2 - \left(\frac{85/8}{7/2}\right)x + \left(\frac{7}{2}\right) = 0$

$x^2 - \frac{85}{28}x + \frac{7}{2} = 0$

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Question 4 continued

$x^2 - \frac{85}{28}x + \frac{7}{2} = 0$ $\times 28$

$28x^2 - 85x + 98 = 0$ ← all integer coefficients

$\boxed{28x^2 - 85x + 95 = 0}$

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5. (a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n}(r+1)(r+5)=\frac{n}{6}(n+7)(2n+7)$$

for all positive integers n.

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n}(r+1)(r+5)=\frac{7n}{6}(n+1)(an+b)$$

where a and b are integers to be determined.

(2)

a. $\sum_{r=1}^{n}(r+1)(r+5) = \sum_{r=1}^{n} r^2 + 6r + 5$

split summation into 3.

$\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 6r + \sum_{r=1}^{n} 5$

factor 6 out summation

factor 5 out summation

$= \sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r + 5\sum_{r=1}^{n} 1$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$= \frac{1}{6}n(n+1)(2n+1) + 6\left[\frac{1}{2}n(n+1)\right] + 5\left[n\right]$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{6}{2}n(n+1) + 5n$

$= \frac{1}{6}n(n+1)(2n+1) + 3n(n+1) + 5n$

$= \frac{n}{6}\left[(n+1)(2n+1) + 18(n+1) + 30\right]$

$= \frac{n}{6}\left[2n^2 + 3n + 1 + 18n + 18 + 30\right]$

$= \frac{n}{6}\left[2n^2 + 21n + 49\right]$

$\boxed{\frac{n}{6}(n+7)(2n+7) \text{ // shown}}$

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Question 5 continued

b. $\sum_{r=n+1}^{2n}(r+1)(r+5) = \sum_{r=1}^{2n}(r+1)(r+5) - \sum_{r=1}^{n}(r+1)(r+5)$

$= \frac{2n}{6}(2n+7)(2(2n)+7) - \frac{n}{6}(n+7)(2n+7)$

$= \frac{n}{3}(2n+7)(4n+7) - \frac{n}{6}(n+7)(2n+7)$

factor out $\frac{7}{6}n(2n+7)$

$= \frac{7n}{6}(2n+7)\left[\frac{2}{7}(4n+7) - \frac{1}{7}(n+7)\right]$

$= \frac{7n}{6}(2n+7)\left[\frac{8n}{7} + 2 - \frac{1}{7}n - 1\right]$

$= \frac{7}{6}n(2n+7)(n+1)$

$\boxed{\frac{7}{6}n(n+1)(2n+7) \quad a=2, b=7}$

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6. The complex number z is defined by

$$z = -\lambda + 3\text{i} \quad \text{where } \lambda \text{ is a positive real constant}$$

Given that the modulus of z is 5

(a) write down the value of λ

(1)

(b) determine the argument of z, giving your answer in radians to one decimal place.

(2)

In part (c) you must show detailed reasoning.

Solutions relying on calculator technology are not acceptable.

(c) Express in the form $a + \text{i}b$ where a and b are real,

(i) $\dfrac{z + 3\text{i}}{2 - 4\text{i}}$

(ii) z^2

(5)

(d) Show on a single Argand diagram the points A, B, C and D that represent the complex numbers

$$z,\ z^*,\ \frac{z + 3\text{i}}{2 - 4\text{i}} \text{ and } z^2$$

(3)

a. $z = -\lambda + 3i$

$|z| = 5$

$\sqrt{(-\lambda)^2 + (3)^2} = 5$

$(-\lambda)^2 + (3)^2 = (5)^2$

$\lambda^2 + 9 = 25$

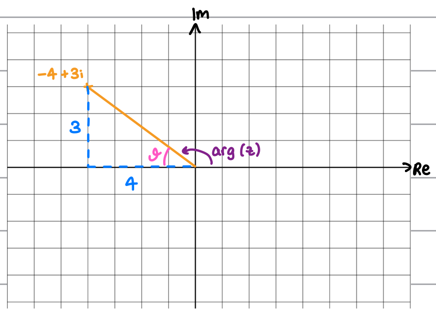
$\lambda^2 = 16$

$\lambda = \pm\sqrt{16} = \pm 4$

However, Q states λ is a real positive constant $\therefore \lambda = 4$ only

$\boxed{\lambda = 4}$

b.



$z = -4 + 3i$

$\theta = \tan^{-1}\left(\frac{3}{4}\right)$

$\arg(z) = \pi - \tan^{-1}\left(\frac{3}{4}\right)$

$\arg(z) = 2.498091545^c$

$\boxed{\arg(z) = 2.5^c \ (1\text{d.p.})}$

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Question 6 continued

c i. $\frac{z+3i}{2-4i} = \frac{(-4+3i)+3i}{2-4i} = \frac{-4+6i}{2-4i}$

must change $\frac{-4+6i}{2-4i}$ into $a+bi$ form:

multiply top and bottom by complex conjugate of denominator

$\frac{-4+6i}{2-4i} \times \frac{2+4i}{2+4i} = \frac{(-4+6i)(2+4i)}{(2-4i)(2+4i)}$

$= \frac{-8-16i+12i+24i^2}{4+8i-8i-16i^2}$

$= \frac{-8+24i^2-4i}{4-16i^2}$

$= \frac{-8+24(-1)-4i}{4-16(-1)}$

$= \frac{-32-4i}{20} = \frac{-32}{20} - \frac{4}{20}i = -\frac{8}{5} - \frac{1}{5}i$

$\boxed{-\frac{8}{5} - \frac{1}{5}i}$

ii. $z^2 = (-4+3i)^2$

$= 16 - 12i - 12i + 9i^2$

$= 16 + 9i^2 - 24i$

$= 16 + 9(-1) - 24i$

$= 7 - 24i$

$\boxed{7-24i}$

d. z^* is the complex conjugate of z.

(flip sign of imaginary component

$\boxed{z^* = -4-3i}$

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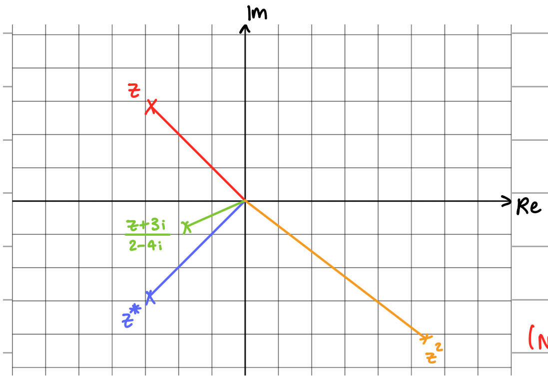
Question 6 continued



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7. The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix}$$

The transformation represented by **A** maps triangle T onto triangle T'

Given that the area of triangle T is 23 cm^2

(a) determine the area of triangle T'

(2)

The point P has coordinates $(3p + 2, 2p - 1)$ where p is a constant. The transformation represented by **A** maps P onto the point P' with coordinates $(17, -18)$

(b) Determine the value of p.

(2)

Given that

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c) describe fully the single geometrical transformation represented by matrix **B**

(2)

The transformation represented by matrix **A** followed by the transformation represented by matrix **C** is equivalent to the transformation represented by matrix **B**

(d) Determine **C**

(3)

a. $|\det(A)|$ = area scale factor

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(A) = (4)(2) - (-5)(-3)$

$= -7$

$\det(A) = -7$

Area S.F. $= |\det(A)| = |-7| = 7$

Area of T $\times$ 7 = Area of T'

23 $\times$ 7 = 161

$\boxed{\text{Area of } T' = 161 \text{ units}^2}$

Question 7 continued

b. $\begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3p+2 \\ 2p-1 \end{bmatrix} = \begin{bmatrix} 17 \\ -18 \end{bmatrix}$

$\begin{bmatrix} 4(3p+2) + -5(2p-1) \\ -3(3p+2) + 2(2p-1) \end{bmatrix} = \begin{bmatrix} 17 \\ -18 \end{bmatrix}$

$\begin{bmatrix} 12p + 8 - 10p + 5 \\ -9p - 6 + 4p - 2 \end{bmatrix} = \begin{bmatrix} 17 \\ -18 \end{bmatrix}$

$\begin{bmatrix} 2p + 13 \\ -5p - 8 \end{bmatrix} = \begin{bmatrix} 17 \\ -18 \end{bmatrix}$

$2p + 13 = 17$ $\qquad$ $-5p - 8 = -18$

$2p = 4$ $\qquad$ $-5p = -10$

$p = 2$ $\qquad$ $p = 2$

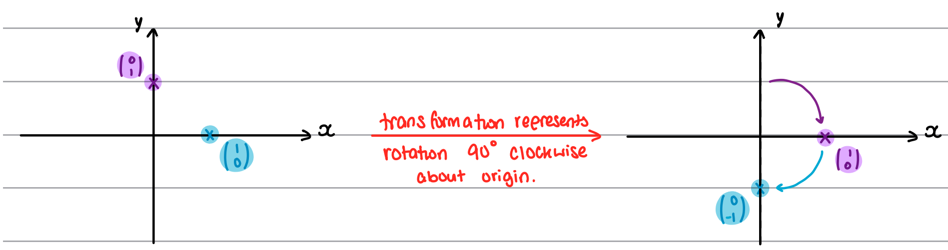
$\boxed{p = 2}$

c. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then mark the coordinates of new matrix

identity matrix I $\qquad$ matrix B

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$\boxed{\text{Rotation, through } 90^\circ \text{ clockwise about origin}}$

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Question 7 continued

d. order: $CA = B$

$CA = B$) $\times A^{-1}$ on right

$C\cancel{AA^{-1}} = BA^{-1}$

$C = BA^{-1}$

(1) calculate A^{-1} first

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(x) = (a)(d) - (b)(c)$

$\det(A) = -7$ from part (a)

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X^{-1} = \frac{1}{\det(x)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(-5) = 5$

$-c = -(-3) = 3$

$A^{-1} = -\frac{1}{7} \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times -\frac{1}{7} \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$C = -\frac{1}{7} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$C = -\frac{1}{7} \begin{bmatrix} (0)(2) + (1)(3) & (0)(5) + (1)(4) \\ (-1)(2) + (0)(3) & (-1)(5) + (0)(4) \end{bmatrix}$

$\boxed{C = -\frac{1}{7} \begin{bmatrix} 3 & 4 \\ -2 & -5 \end{bmatrix}}$

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8. The hyperbola H has Cartesian equation $xy = 25$

The parabola P has parametric equations $x = 10t^2$, $y = 20t$

The hyperbola H intersects the parabola P at the point A

(a) Use algebra to determine the coordinates of A

(3)

The point B with coordinates $(10, 20)$ lies on P

(b) Find an equation for the normal to P at B

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers to be determined.

(5)

(c) Use algebra to determine, in simplest form, the exact coordinates of the points where this normal intersects the hyperbola H

(6)

a. Firstly rewrite Parabola eq^n in terms of cartesian.

$(at^2, 2at) \equiv (10t, 20t^2) \Rightarrow a = 10$

$y^2 = 4ax \Rightarrow y^2 = 4(10)x$

$y^2 = 40x$ ☆

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	~~$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$~~	$y^2 = 4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	~~$xy = c^2$~~
Parametric Form	~~$(a\cos\theta, b\sin\theta)$~~	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	~~$\left(ct, \frac{c}{t}\right)$~~
Eccentricity	~~$e<1$ $b^2=a^2(1-e^2)$~~	$e = 1$	$e>1$ $b^2=a^2(e^2-1)$	~~$e=\sqrt{2}$~~
Foci	~~$(\pm ae, 0)$~~	$(a, 0)$	$(\pm ae, 0)$	~~$(\pm\sqrt{2}c, \pm\sqrt{2}c)$~~
Directrices	~~$x=\pm\frac{a}{e}$~~	$x = -a$	$x=\pm\frac{a}{e}$	~~$x+y=\pm\sqrt{2}c$~~
Asymptotes	~~none~~	none	$\frac{x}{a}=\pm\frac{y}{b}$	~~$x=0, y=0$~~

Can now solve both $y^2 = 40x$ and $xy = 25$ simultaneously.

$xy = 25$ $\times 40$ on both sides

$40(xy) = 40(25) \Rightarrow 40xy = 1000$

$(y^2)y = 1000$ substitute $y^2 = 40x$ into $40xy = 1000$

$y^3 = 1000$

$y = \sqrt[3]{1000} = 10$

$x(10) = 25$

$x = \frac{25}{10} = \frac{5}{2}$

$\boxed{\therefore A\left(\frac{5}{2}, 10\right)}$

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Question 8 continued

b. Differentiate parabola eq^n w.r.t. x:

$y^2 = 40x$

$2y\left(\frac{dy}{dx}\right) = 40$

$\frac{dy}{dx} = \frac{40}{2y} = \frac{20}{y}$

Sub in coord. B $(10, 20)$:

$\left.\frac{dy}{dx}\right|_{(10,20)} \quad \frac{20}{(20)} = 1$

$M_{tangent} = 1$

$M_{normal} = -1$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = -1$

$(x_1, y_1) = (10, 20)$

$y - 20 = -1(x - 10)$

$y - 20 = -x + 10$

$x + y - 30 = 0$

$\boxed{x + y - 30 = 0 \,/\!/ \quad a = 1,\ b = 1,\ c = -30}$

c. Solve $x + y - 30 = 0$ and $xy = 25$ simultaneously.

$x + y - 30 = 0$ rearrange for x

$x = 30 - y$

$xy = 25$ sub $x = 30 - y$ into hyperbola eq^n

$(30 - y)y = 25$

$30y - y^2 = 25$

$y^2 - 30y + 25 = 0$

$y^2 - 30y + 25 = 0$

$(y - 15)^2 - 225 + 25 = 0$ solve for y

$(y - 15)^2 - 200 = 0$

$y = 15 \pm \sqrt{200} = 15 \pm 10\sqrt{2}$

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Question 8 continued

now sub y-coords into either eq^{n}s to find x-coords.

$\Rightarrow$ when $y = 15 + 10\sqrt{2}$

$x = 30 - y$

$x = 30 - (15 + 10\sqrt{2})$

$x = 30 - 15 - 10\sqrt{2}$

$x = 15 - 10\sqrt{2}$

$\Rightarrow$ when $y = 15 - 10\sqrt{2}$

$x = 30 - y$

$x = 30 - (15 - 10\sqrt{2})$

$x = 30 - 15 + 10\sqrt{2}$

$x = 15 + 10\sqrt{2}$

Points of Intersection: $(15 + 10\sqrt{2}, 15 - 10\sqrt{2})$
$(15 - 10\sqrt{2}, 15 + 10\sqrt{2})$

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9. (i) A sequence of numbers $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = \frac{1}{3}(2u_n - 1) \qquad u_1 = 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3\left(\frac{2}{3}\right)^n - 1$$

(6)

(ii) $f(n) = 2^{n+2} + 3^{2n+1}$

Prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is a multiple of 7

(6)

i. (1) Base case: Prove true for $n=1$

$u_1 = 3\left(\frac{2}{3}\right)^1 - 1 = 1$

$u_1 = 1$

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$u_k = 3\left(\frac{2}{3}\right)^k - 1$

(3) Prove true for $n = k+1$:

$u_{k+1} = \frac{1}{3}(2u_k - 1)$

$u_{k+1} = \frac{1}{3}\left(2\left[3\left(\frac{2}{3}\right)^k - 1\right] - 1\right)$

$= \frac{1}{3}\left(6\left(\frac{2}{3}\right)^k - 2 - 1\right)$

$= \frac{1}{3}\left(6\left(\frac{2}{3}\right)^k - 3\right)$

$= \frac{1}{3}\left(2 \times 3\left(\frac{2}{3}\right)^k - 3\right)$

$= \frac{2}{3} \times 3\left(\frac{2}{3}\right)^k - \left(\frac{1}{3}\right)(3)$

$= 3 \times \left(\frac{2}{3}\right)^k \times \left(\frac{2}{3}\right)^1 - 1$

$= 3\left(\frac{2}{3}\right)^{k+1} - 1$

Thinking ahead, sub in $n = k+1$ into u_n - this is what we are trying to prove using (2):

$u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$

Make a note to the side so we know what we are working towards.

$\therefore u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$ true for $n = k+1$

(4) Conclusion

Hence result is true for $n = k+1$. As true for $n=1$ and have shown if true for $n=k$ then it is true for $n = k+1$, so it is true for all $n \in \mathbb{N}$ by induction.

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Question 9 continued

ii. (1) Base Case: check true for $n=1$

$f(1) = 2^{1+2} + 3^{2(1)+1}$

$= 35$

$= 5(7)$ ← divisible by 7

∴ true for $n=1$

(2) Assume true for $n=k$:

$f(k) = 2^{k+2} + 3^{2k+1}$

(3) Inductive Step, Prove true for $n=k+1$:

$f(k+1) = 2^{k+1+2} + 3^{2(k+1)+1}$

$= 2^{k+3} + 3^{2k+2+1}$

$= 2^{k+3} + 3^{2k+3}$

$f(k+1) - f(k) = (2^{k+3} + 3^{2k+3}) - (2^{k+2} + 3^{2k+1})$

$= [2(2^{k+2}) + 3^2(3^{2k+1})] - (2^{k+2} + 3^{2k+1})$

$= 2(2^{k+2}) + 9(3^{2k+1}) - 2^{k+2} - 3^{2k+1}$

$= 2^{k+2} + 8(3^{2k+1})$

$= 2^{k+2} + 3^{2k+1} + 7(3^{2k+1})$

$= f(k) + 7(3^{2k+1})$

$f(k+1) - f(k) = f(k) + 7(3^{2k+1})$

$f(k+1) = 2f(k) + 7(3^{2k+1})$

∴ $f(k+1) = 2f(k) + 7(3^{2k+1})$ ∴ true for $n=k+1$

(4) Conclusion

Hence result is true for $n=k+1$. As true for $n=1$ and have shown if true for $n=k$ then it is true for $n=k+1$, so it is true for all $n \in \mathbb{N}$ by induction.

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